On $\mathrm{SL}(3,\mathbf{C})$-covariant spinor equation and generalized Duffin-Kemmer algebra

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Abstract

The $\mathrm{SL}(3,\mathbf{C})$-covariant 9-dimensional equation for a free 3-spinor particle is transformed into the Dirac-like form $(p_A\delta^A - M)\Psi = 0$. However, the corresponding δ matrices do not satisfy the Dirac algebra. It is shown that δ^A lead to a Finslerian generalization of the Duffin-Kemmer algebra. The Appendix contains an explicit realization of the δ matrices.

During several years, a theory called *binary geometrophysics* is being successfully developed [1]. Within the framework of this theory, in the article [2], the concept of a *3-spinor* as a tensor on the complex vector space $\mathbf{C}^3$ with the trilinear antisymmetric "scalar multiplication" $\mathbf{C}^3\times\mathbf{C}^3\times\mathbf{C}^3\to\mathbf{C}$ was introduced. Isometries of such a space form a group isomorphic to the group $\mathrm{SL}(3,\mathbf{C})$ of unimodular complex $3{\times}3$ matrices. In the same article, a homomorphism of $\mathrm{SL}(3,\mathbf{C})$ into the isometry group of a 9-dimensional (over $\mathbf{R}$) flat Finslerian space with the metric

$$\begin{aligned}G^{ABC}p_Ap_Bp_C &= (p_0^2-p_1^2-p_2^2-p_3^2)p_8\\ &\quad -(p_4^2+p_5^2+p_6^2+p_7^2)p_0+2(p_4p_6+p_5p_7)p_1\\ &\quad +2(p_5p_6-p_4p_7)p_2+(p_4^2+p_5^2-p_6^2-p_7^2)p_3 \qquad (1)\end{aligned}$$

was constructed. Here $A, B, C = \overline{0,8}$ and p_A is a real 9-vector. It should be noted that (1) passes into the ordinary pseudoeuclidean 4-metric $g^{\mu\nu}p_\mu p_\nu = p_0^2 - p_1^2 - p_2^2 - p_3^2$ after the procedure of the dimensional reduction. From the physical point of view, the expression (1) is the "scalar cube" of a vector in the 9-dimensional momentum space.

Let i^r and $\beta_{\dot s}$ $(r, s = \overline{1,3})$ be 3-spinors of rank one while $P^{r\dot s}$ be a 3-spinor of rank two such that the matrix $P \equiv \|P^{r\dot s}\|$ is Hermitian. (As ever, the dotted index means that the corresponding quantity is transformed according to the complex conjugate representation of the group $\mathrm{SL}(3, \mathbf{C})$.) It is not difficult to show that

$$\det P = G^{ABC} p_A p_B p_C \tag{2}$$

when the relations between $P^{r\dot s}$ and p_A have the form

$$\left.\begin{array}{l} P^{1\dot 1} = p_0 + p_3,\ \ P^{1\dot 2} = p_1 - ip_2,\ P^{1\dot 3} = p_4 - ip_5 \\ P^{2\dot 1} = p_1 + ip_2,\ P^{2\dot 2} = p_0 - p_3,\ \ P^{2\dot 3} = p_6 - ip_7 \\ P^{3\dot 1} = p_4 + ip_5,\ P^{3\dot 2} = p_6 + ip_7,\ P^{3\dot 3} = p_8 \end{array}\right\}. \tag{3}$$

In the work [3], to describe a free 3-spinor particle with a 9-momentum p_A, the $\mathrm{SL}(3, \mathbf{C})$-covariant equation

$$\left.\begin{array}{l} P^{r\dot s}\beta_{\dot s} = Mi^r \\ P_{r\dot s}\, i^r = M^2\beta_{\dot s} \end{array}\right\} \tag{4}$$

was proposed, where $P^{r\dot s}$ are defined by (3), M is a real scalar, and $P_{r\dot s}$ are the cofactors of $P^{r\dot s}$. It is natural to call M the 9-mass of a particle because substituting the upper equality of (4) into the lower one (and vice versa) gives a Finslerian analog of the Klein-Gordon equation for each 3-spinor component: $(G^{ABC}p_A p_B p_C - M^3)i^r = 0$, $(G^{ABC}p_A p_B p_C - M^3)\beta_{\dot s} = 0$. In [3], it was also shown that (4) split into the standard 4-dimensional Dirac and Klein-Gordon equations after the group reduction $\mathrm{SL}(3, \mathbf{C}) \to \mathrm{SL}(2, \mathbf{C})$.

In general, the equation (4) is *quadratic* with respect to p_A. This follows from (3) and the fact that $P_{r\dot s}$ are proportional to 2×2 minors of the matrix P. The purpose of the present paper is to transform (4) into the form of an equation *linear* with respect to the 9-momentum p_A. To this end, it is possible to use the method of Duffin and Kemmer [4, 5].

Let us introduce the new variables ξ_1, ξ_2, …, ξ_6 such that

$$\left.\begin{array}{ll} P^{2\dot{1}}i^1 - P^{1\dot{1}}i^2 = M\xi_1, & P^{2\dot{2}}i^1 - P^{1\dot{2}}i^2 = M\xi_4 \\ P^{3\dot{1}}i^1 - P^{1\dot{1}}i^3 = M\xi_2, & P^{3\dot{2}}i^1 - P^{1\dot{2}}i^3 = M\xi_5 \\ P^{3\dot{1}}i^2 - P^{2\dot{1}}i^3 = M\xi_3, & P^{3\dot{2}}i^2 - P^{2\dot{2}}i^3 = M\xi_6 \end{array}\right\}. \tag{5}$$

With the help of (5), one can rewrite the lower equality of (4) in the following form

$$\left.\begin{array}{r} P^{3\dot{3}}\xi_4 - P^{2\dot{3}}\xi_5 + P^{1\dot{3}}\xi_6 = M\beta_{\dot{1}} \\ -P^{3\dot{3}}\xi_1 + P^{2\dot{3}}\xi_2 - P^{1\dot{3}}\xi_3 = M\beta_{\dot{2}} \\ -P^{3\dot{1}}\xi_4 + P^{2\dot{1}}\xi_5 - P^{1\dot{1}}\xi_6 = M\beta_{\dot{3}} \end{array}\right\}. \tag{6}$$

Thus, (4) is equivalent to the set of equations $P^{r\dot{s}}\beta_{\dot{s}} = Mi^r$, (5)–(6) or, what is the same, to the matrix equation

$$\hat{P}\Psi = M\Psi, \tag{7}$$

where $\Psi = (i^1, i^2, i^3, \beta_{\dot{1}}, \beta_{\dot{2}}, \beta_{\dot{3}}, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)^\top$ is the 12-component column and

$$\hat{P} = \begin{pmatrix} \mathbf{0} & P & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & P_1 & P_2 \\ P_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ P_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{8}$$

is the 12×12 matrix consisting of the 3×3-blocks:

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{9}$$

$$P = \begin{pmatrix} P^{1\dot{1}} & P^{1\dot{2}} & P^{1\dot{3}} \\ P^{2\dot{1}} & P^{2\dot{2}} & P^{2\dot{3}} \\ P^{3\dot{1}} & P^{3\dot{2}} & P^{3\dot{3}} \end{pmatrix}, \tag{10}$$

$$P_1 = \begin{pmatrix} 0 & 0 & 0 \\ -P^{3\dot{3}} & P^{2\dot{3}} & -P^{1\dot{3}} \\ 0 & 0 & 0 \end{pmatrix}, \tag{11}$$

$$P_2 = \begin{pmatrix} P^{3\dot{3}} & -P^{2\dot{3}} & P^{1\dot{3}} \\ 0 & 0 & 0 \\ -P^{3\dot{1}} & P^{2\dot{1}} & -P^{1\dot{1}} \end{pmatrix}, \tag{12}$$

$$P_3 = \begin{pmatrix} P^{2\dot{1}} & -P^{1\dot{1}} & 0 \\ P^{3\dot{1}} & 0 & -P^{1\dot{1}} \\ 0 & P^{3\dot{1}} & -P^{2\dot{1}} \end{pmatrix}, \tag{13}$$

$$P_4 = \begin{pmatrix} P^{2\dot{2}} & -P^{1\dot{2}} & 0 \\ P^{3\dot{2}} & 0 & -P^{1\dot{2}} \\ 0 & P^{3\dot{2}} & -P^{2\dot{2}} \end{pmatrix}. \tag{14}$$

Let us raise (8) to the fourth power. The direct calculation shows that

$$\hat{P}^4 = (\det P)\hat{P}. \tag{15}$$

On the other hand, by using (3) and (9)–(14), it is possible to represent (8) in the form of the linear combination

$$\hat{P} = p_{A}\delta^{A} \tag{16}$$

of nine 12×12 matrices δ^{A} which are given in the Appendix. The substitution of (2) and (16) into (15) results in the identity

$$(p_{A}\delta^{A})^4 = G^{ABC}p_{A}p_{B}p_{C}(p_{D}\delta^{D}) \tag{17}$$

which is valid for any p_{A}; here $A, B, C, D = \overline{0,8}$. It is evident that (17) generalizes the known 4-dimensional identity $(p_\mu\beta^\mu)^3 = g^{\mu\nu}p_\mu p_\nu(p_\lambda\beta^\lambda)$, where $\mu, \nu, \lambda = \overline{0,3}$ and β^μ are the Duffin-Kemmer matrices [4, 5]. Moreover, it follows from (17) that the δ matrices satisfy the conditions

$$\delta^{(A}\delta^{B}\delta^{C}\delta^{D)} = 6\{G^{ABC}\delta^{D} + G^{ABD}\delta^{C} + G^{ACD}\delta^{B} + G^{BCD}\delta^{A}\}, \tag{18}$$

where the parentheses denote the symmetrization with respect to all the superscripts (i.e., the sum over all permutations of A, B, C, D). In this connection, it is interesting to recall important relations of the Duffin-Kemmer algebra

$$\beta^{(\mu}\beta^{\nu}\beta^{\lambda)} = 2\{g^{\mu\nu}\beta^{\lambda} + g^{\lambda\mu}\beta^{\nu} + g^{\lambda\nu}\beta^{\mu}\}. \tag{19}$$

It is easy to see the full analogy between the formulae (18) and (19).

Let us return to the equation (7). With the help of (16), it can be written finally in the Dirac-like form

$$(p_A \delta^A - M)\Psi = 0, \tag{20}$$

where δ^A satisfy the conditions (18). Thus, the purpose of this paper has been achieved: (4) has been transformed into the p-linear equation (20).

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Appendix. The explicit form of the δ matrices

$$\delta^0 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$\delta^1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$\delta^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\delta^3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\delta^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\delta^5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\delta^6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\delta^7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\delta^8 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

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